MULTISYMPLECTIC LAGRANGIAN AND HAMILTONIAN FORMALISMS OF FIRST-ORDER CLASSICAL FIELD THEORIES

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Abstract

This review paper is devoted to presenting the standard *multisymplectic formulation* for describing geometrically first-order classical field theories, both the regular and singular cases. First, the main features of the Lagrangian formalism are revisited and, second, the Hamiltonian formalism is constructed using *Hamiltonian sections*. In both cases, the variational principles leading to the Euler-Lagrange and the Hamilton-De Donder-Weyl equations, respectively, are stated, and these field equations are given in different but equivalent geometrical ways in each formalism. Finally, both are unified in a new formulation (which has been recently developed), following the original ideas of Rusk and Skinner for mechanical systems.

Key words: *First-order field theories, Lagrangian and Hamiltonian formalisms, Fiber bundles, Multisymplectic manifolds.*

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Contents

1 Introduction

In recent years much work has been done with the aim of establishing the suitable geometrical structures for describing classical field theories.

There are different kinds of geometrical models for making a covariant description of first-order field theories. For instance, we have the so-called *k-symplectic formalism* which uses the *k-symplectic forms* introduced by Awane [4, 5, 6], and which coincides with the *polysymplectic formalism* described by Günther [36] (see also [63]). A natural extension of this is the *k-cosymplectic formalism*, which is the generalization to field theories of the cosymplectic description of non-autonomous mechanical systems [57, 58]. Furthermore, the polysymplectic formalism developed by Sardanashvily *et al* [29, 68] and Kanatchikov [41], based on the use of a vector-valued form on a fiber bundle, is a different description of classical field theories than the polysymplectic one proposed by Günther. In addition, soldering forms on linear frame bundles are also polysymplectic forms, and their study and applications to field theory constitute the k-symplectic geometry developed by Norris [64, 65, 66]. There also exists the formalism based on using *Lepagean forms*, used for describing certain kinds of equivalent Lagrangian models with non-equivalent Hamiltonian descriptions [47, 48, 49, 50]. Finally, a new geometrical framework for field theories based on the use of Lie algebroids has been developed in a recent work [62].

In this work, we consider only the *multisymplectic* models [13, 31, 33, 51, 61], first introduced by Kijowski and Tulkcyjew [42, 43, 44]. They arise from the study of multisymplectic manifolds and their properties (see [9, 10] for recent references, and Appendix A.1 for a brief review); in particular, those concerning the behavior of multisymplectic Lagrangian and Hamiltonian systems.

The usual way of working with field theories consists in stating their Lagrangian formalism [3, 8, 12, 20, 21, 27, 29, 30, 69], and jet bundles are the appropriate domain for doing so. The construction of this formalism for regular and singular theories is reviewed in Section 2.

The Hamiltonian description presents different kinds of problems. For instance, the choice of the multimomentum bundle for developing the theory is not unique [23], and different kinds of Hamiltonian systems can be defined, depending on this choice and on the way of introducing the physical content (the "Hamiltonian") [17, 19, 37, 38, 60, 67]. Here we present one of the most standard ways of defining Hamiltonian systems, which is based on using *Hamiltonian sections* [11]; although this construction can also be done taking *Hamiltonian densities* [11, 29, 61, 68]. In particular, the construction of Hamiltonian systems which are the Hamiltonian counterpart of Lagrangian systems is carried out by using the *Legendre map* associated with the Lagrangian system, and this problem has been studied by different authors in the *(hyper) regular* case [11, 69], and in the *singular* (*almost-regular*) case [29, 52, 68]. In Section 3 we review these constructions.

Another subject of interest in the geometrical description of Classical Field theories concerns the field equations. In the multisymplectic models, both in the Lagrangian and Hamiltonian formalisms, these equations can be derived from a suitable variational principle: the so-called *Hamilton principle* in the Lagrangian formalism and *Hamilton-Jacobi principle* in the Hamiltonian formulation [3, 17, 20, 23, 27, 30], and the field equations are usually written by using the multisymplectic form in order to characterize the critical sections which are solutions of the problem. In addition, these critical sections can be thought of as being the integral manifolds of certain kinds of integrable multivector fields or Ehresmann connections, defined in the bundles where the formalism is developed, and satisfying a suitable geometric equation which is the intrinsic formulation of the systems of partial differential equations locally describing the field [20, 21, 22, 52, 69]. All these aspects are discussed in Sections 2 and 3 (furthermore, a quick review on multivector fields and connections is given in Appendix A.2). Moreover, multivector fields are also used in order to state generalized Poisson brackets in the Hamiltonian formalism of field theories [26, 39, 40, 41, 67].

In ordinary mechanics there is also a unified formulation of Lagrangian and Hamiltonian formalisms [71], which is based on the use of the *Whitney sum* of the tangent and cotangent bundles (the *velocity* and *momentum phase spaces* of the system). This formalism has been generalized for non-autonomous mechanics [14, 35] and recently for first-order classical field theories [18, 54]. The main features of this formulation are explained in Section 4.

Finally, we ought to point out that there are also geometric frameworks for describing the non-covariant or space-time formalism of field theories, where the use of *Cauchy surfaces* is the fundamental tool [32, 34, 56]. Furthermore, in recent years, numerical methods have been developed for solving the field equations, which are based on the use of *multisymplectic integrators* [59, 61]. As a final remark, many of the above subjects have also been studied for higher-order field theories (see, for instance, [1, 2, 24, 25, 28, 45, 46, 69, 70]). Nevertheless we do not consider these topics in this survey.

In this paper, manifolds are real, paracompact, connected and C^∞, maps are C^∞, and sum over crossed repeated indices is understood.

2 Lagrangian formalism

2.1 Lagrangian systems

A *first-order classical field theory* is described by the following elements: First, we have the *configuration fibre bundle* $\pi\colon E \to M$, with $\dim M = m$ and $\dim E = n + m$, where M is an oriented manifold with volume form $\omega \in \mathit{\Omega}^m(M)$. $\pi^1\colon J^1\pi \to E$ is the first-order jet bundle of local sections of π, which is also a bundle over M with projection $\bar{\pi}^1 = \pi \circ \pi^1\colon J^1\pi \longrightarrow M$, and $\dim J^1\pi = nm+n+m$. We denote by (x^ν, y^A, v^A_ν) ($\nu = 1,\dots,m$; $A = 1,\dots,n$) natural coordinates in $J^1\pi$ adapted to the bundle structure and such that $\omega = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m \equiv \mathrm{d}^m x$. Second, we give the *Lagrangian density*, which is a $\bar{\pi}^1$-semibasic m-form on $J^1\pi$ and hence it can be expressed as $\mathcal{L} = \pounds(\bar{\pi}^{1*}\omega)$, where $\pounds \in \mathrm{C}^\infty(J^1\pi)$ is the *Lagrangian function* associated with $\mathcal{L}$ and ω.

$J^1\pi$ is endowed with a canonical structure, $\mathcal{V} \in \Omega^1(J^1\pi) \otimes \Gamma(J^1\pi, \mathrm{V}(\pi^1)) \otimes \Gamma(J^1\pi, \bar{\pi}^{1^*}\mathrm{T}M)$, which is called the *vertical endomorphism* [20, 27, 30, 69] (here $\mathrm{V}(\pi^1)$ denotes the vertical subbundle with respect to the projection π^1, and $\Gamma(J^1\pi, \mathrm{V}(\pi^1))$ the set of sections in the corresponding bundle). Then the *Poincaré-Cartan m and $(m+1)$-forms* associated with $\mathcal{L}$ are defined as

$$\Theta_{\mathcal{L}} := i(\mathcal{V})\mathcal{L} + \mathcal{L} \in \Omega^m(J^1\pi) \quad ; \quad \Omega_{\mathcal{L}} := -\mathrm{d}\Theta_{\mathcal{L}} \in \Omega^{m+1}(J^1\pi)$$

We have the following local expressions (where $\mathrm{d}^{m-1}x_\alpha \equiv i\left(\dfrac{\partial}{\partial x^\alpha}\right)\mathrm{d}^m x$):

$$\begin{aligned}
\Theta_{\mathcal{L}} &= \frac{\partial \pounds}{\partial v^A_\nu}\mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu - \left(\frac{\partial \pounds}{\partial v^A_\nu}v^A_\nu - \pounds\right)\mathrm{d}^m x \\
\Omega_{\mathcal{L}} &= -\frac{\partial^2 \pounds}{\partial v^B_\nu \partial v^A_\alpha}\mathrm{d}v^B_\nu \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\alpha - \frac{\partial^2 \pounds}{\partial y^B \partial v^A_\alpha}\mathrm{d}y^B \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\alpha + \\
&\quad \frac{\partial^2 \pounds}{\partial v^B_\nu \partial v^A_\alpha}v^A_\alpha \mathrm{d}v^B_\nu \wedge \mathrm{d}^m x + \left(\frac{\partial^2 \pounds}{\partial y^B \partial v^A_\alpha}v^A_\alpha - \frac{\partial \pounds}{\partial y^B} + \frac{\partial^2 \pounds}{\partial x^\alpha \partial v^B_\alpha}\right)\mathrm{d}y^B \wedge \mathrm{d}^m x
\end{aligned}$$

Definition 1 $(J^1\pi, \Omega_{\mathcal{L}})$ *is said to be a* Lagrangian system.

The Lagrangian system and the Lagrangian function are said to be regular *if $\Omega_{\mathcal{L}}$ is a multisymplectic $(m+1)$-form (i.e.; 1-nondegenerate). Elsewhere they are* singular *(or* non-regular*).*

The regularity condition is locally equivalent to $det\left(\dfrac{\partial^2 \pounds}{\partial v^A_\alpha \partial v^B_\nu}(\bar{y})\right) \neq 0$, $\forall \bar{y} \in J^1\pi$. We must point out that, in field theories, the notion of regularity is not uniquely defined (for other approaches see, for instance, [7, 15, 16, 47, 49, 50]).

2.2 Lagrangian field equations

The Lagrangian field equations can be derived from a variational principle. In fact:

Definition 2 *Let $(J^1\pi, \Omega_{\mathcal{L}})$ be a Lagrangian system. Let $\Gamma(M,E)$ be the set of sections of π. Consider the map*

$$\begin{array}{rccc} \mathbf{L} & : & \Gamma(M,E) & \longrightarrow & \mathbb{R} \\ & & \phi & \mapsto & \int_M (j^1\phi)^*\Theta_{\mathcal{L}} \end{array}$$

(where the convergence of the integral is assumed). The variational problem *for this Lagrangian system is the search of the critical (or stationary) sections of the functional $\mathbf{L}$, with respect to the variations of ϕ given by $\phi_t = \sigma_t \circ \phi$, where $\{\sigma_t\}$ is a local one-parameter group of any compact-supported $Z \in \mathfrak{X}^{\mathrm{V}(\pi)}(E)$ (the module of π-vertical vector fields in E), that is:*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_M (j^1\phi_t)^*\Theta_{\mathcal{L}} = 0$$

This is the Hamilton principle *of the Lagrangian formalism.*

The Hamilton principle is equivalent to find a distribution $\mathcal{D}$ in $J^1\pi$ such that:

1. $\mathcal{D}$ is m-dimensional.

2. $\mathcal{D}$ is $\bar{\pi}^1$-transverse.

3. $\mathcal{D}$ is integrable (that is, *involutive*).

4. The integral manifolds of $\mathcal{D}$ are the canonical liftings to $J^1\pi$ of the critical sections of the Hamilton principle.

A distribution $\mathcal{D}$ satisfying 1 and 2 is associated with a connection in the bundle $\bar{\pi}^1 \colon J^1\pi \to M$ (integrable if 3 holds), whose local expression is

$$\nabla = \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\nu} + F_\nu^A \frac{\partial}{\partial y^A} + G_{\nu\rho}^A \frac{\partial}{\partial v_\rho^A} \right)$$

Furthermore, these kinds of integrable distributions and the corresponding connections are associated with classes of integrable (i.e., non-vanishing, locally decomposable and involutive) $\bar{\pi}^1$-transverse m-multivector fields in $J^1\pi$ (see Appendix A.2). If 2 holds, the local expression in natural coordinates of an element of one of these classes is

$$\mathcal{X} = \bigwedge_{\nu=1}^{m} f \left(\frac{\partial}{\partial x^\nu} + F_\nu^A \frac{\partial}{\partial y^A} + G_{\nu\rho}^A \frac{\partial}{\partial v_\rho^A} \right) \quad , \quad (f \in \mathrm{C}^\infty(J^1\pi) \text{ non-vanishing})$$

If, in addition, the integral sections are holonomic (that is, they are canonical liftings of sections of $\pi \colon E \to M$), then the integrable connections and their associated classes of multivector fields are called *holonomic*. To be holonomic is equivalent to be integrable and *semi-holonomic*, that is, $F_\nu^A = v_\nu^A$ in the above local expressions. Then:

Theorem 1 *Let $(J^1\pi, \Omega_{\mathcal{L}})$ be a Lagrangian system. The following assertions on a section $\phi \in \Gamma(M,E)$ are equivalent:*

1. *ϕ is a critical section for the variational problem posed by the Hamilton principle.*

2. *$(j^1\phi)^*\, i(X)\Omega_{\mathcal{L}} = 0$, $\forall\, X \in \mathfrak{X}(J^1\pi)$.*

3. *$j^1\phi$ is an integral section of a class of holonomic multivector fields $\{\mathcal{X}_{\mathcal{L}}\} \subset \mathfrak{X}^m(J^1\pi)$ satisfying*
$$i(\mathcal{X}_{\mathcal{L}})\Omega_{\mathcal{L}} = 0 \quad , \quad \forall\, \mathcal{X}_{\mathcal{L}} \in \{\mathcal{X}_{\mathcal{L}}\} \tag{1}$$

4. *$j^1\phi$ is an integral section of a holonomic connection $\nabla_{\mathcal{L}}$ in $J^1\pi$ satisfying*
$$i(\nabla_{\mathcal{L}})\Omega_{\mathcal{L}} = (m-1)\Omega_{\mathcal{L}} \tag{2}$$

5. *If $(U; x^\nu, y^A, v_\nu^A)$ is a natural system of coordinates in $J^1\pi$, then $j^1\phi = \left(x^\nu, y^A(x^\eta), \frac{\partial y^A}{\partial x^\nu}(x^\eta) \right)$ in U satisfies the* Euler-Lagrange equations
$$\frac{\partial \pounds}{\partial y^A} \circ j^1\phi - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \pounds}{\partial v_\mu^A} \circ j^1\phi \right) = 0 \quad , \quad (\text{for } A = 1, \ldots, n)$$

(*Proof*) See, for instance [20, 21, 27, 30, 52, 69]. ∎

Semi-holonomic (not necessarily integrable) locally decomposable multivector fields and connections which are solution to the Lagrangian equations (1) and (2) respectively are called *Euler-Lagrange multivector fields* and *connections* for $(J^1\pi, \Omega_{\mathcal{L}})$.

If $(J^1\pi, \Omega_{\mathcal{L}})$ is regular, Euler-Lagrange m-vector fields and connections exist in $J^1\pi$, although they are not necessarily integrable. If $(J^1\pi, \Omega_{\mathcal{L}})$ is singular, in the most favourable cases, these Euler-Lagrange multivector fields and connections only exist in some submanifold $S \hookrightarrow J^1\pi$, which can be obtained after applying a suitable constraint algorithm (see [53]).

3 Hamiltonian formalism

3.1 Multimomentum bundles. Legendre maps

As we have pointed out in the introduction, there are different bundles where the Hamiltonian formalism of field theories can be developed. Here we take one of the most standard choices.

First, $\mathcal{M}\pi \equiv \Lambda_2^m \mathrm{T}^*E$, is the bundle of m-forms on E vanishing by the action of two π-vertical vector fields (so $\dim \mathcal{M}\pi = nm+n+m+1$), and is diffeomorphic to the set $\mathrm{Aff}(J^1\pi, \Lambda^m\mathrm{T}^*M)$, made of the affine maps from $J^1\pi$ to $\Lambda^m\mathrm{T}^*M$ (the multitangent bundle of M of order m) [11], [23]. It is called the *extended multimomentum bundle*, and its canonical submersions are denoted

$$\kappa\colon \mathcal{M}\pi \to E \quad ; \quad \bar{\kappa} = \pi\circ\kappa\colon \mathcal{M}\pi \to M$$

As $\mathcal{M}\pi$ is a subbundle of $\Lambda^m\mathrm{T}^*E$ (the multicotangent bundle of E of order m [10]), then $\mathcal{M}\pi$ is endowed with a canonical form $\Theta \in \mathit{\Omega}^m(\mathcal{M}\pi)$ (the "tautological form"), which is defined as follows: let $(x,\alpha) \in \Lambda_2^m\mathrm{T}^*E$, with $x \in E$ and $\alpha \in \Lambda_2^m\mathrm{T}_x^*E$; then, for every $X_1,\ldots,X_m \in \mathrm{T}_{(x,\alpha)}(\mathcal{M}\pi)$,

$$\Theta((x,\alpha);X_1,\ldots,X_m) := \alpha(x;\mathrm{T}_{(x,\alpha)}\kappa(X_1),\ldots,\mathrm{T}_{(q,\alpha)}\kappa(X_m))$$

Then we define the multisymplectic form $\Omega := -\mathrm{d}\Theta \in \mathit{\Omega}^{m+1}(\mathcal{M}\pi)$. They are known as the *multimomentum Liouville m and $(m+1)$-forms*

If introduce natural coordinates (x^ν, y^A, p_A^ν, p) in $\mathcal{M}\pi$ adapted to the bundle $\pi\colon E\to M$, and such that $\omega = \mathrm{d}^m x$, the local expressions of these forms are

$$\Theta = p_A^\nu \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu + p\mathrm{d}^m x \quad , \quad \Omega = -\mathrm{d}p_A^\nu \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu - \mathrm{d}p\wedge \mathrm{d}^m x$$

Now we denote by $J^1\pi^*$ the quotient $\mathcal{M}\pi/\pi^*\Lambda^m\mathrm{T}^*M$, with $\dim J^1\pi^* = nm+n+m$. We have the natural submersions

$$\tau\colon J^1\pi^* \to E \quad ; \quad \bar{\tau} = \pi\circ\tau\colon J^1\pi^* \to M$$

Furthermore, the natural submersion $\mu\colon \mathcal{M}\pi \to J^1\pi^*$ endows $\mathcal{M}\pi$ with the structure of an affine bundle over $J^1\pi^*$, with $(\pi\circ\tau)^*\Lambda^m\mathrm{T}^*M$ as the associated vector bundle. $J^1\pi^*$ is usually called the *restricted multimomentum bundle* associated with the bundle $\pi\colon E\to M$.

Natural coordinates in $J^1\pi^*$ (adapted to the bundle $\pi\colon E\to M$) are denoted by (x^ν, y^A, p_A^ν).

Definition 3 *Let $(J^1\pi, \Omega_{\mathcal{L}})$ be a Lagrangian system. The extended Legendre map associated with $\mathcal{L}$, $\widetilde{\mathcal{FL}}\colon J^1\pi \to \mathcal{M}\pi$, is defined by*

$$(\widetilde{\mathcal{FL}}(\bar{y}))(Z_1,\ldots,Z_m) := (\Theta_{\mathcal{L}})_{\bar{y}}(\bar{Z}_1,\ldots,\bar{Z}_m)$$

where $Z_1,\ldots,Z_m \in \mathrm{T}_{\pi^1(\bar{y})}E$, and $\bar{Z}_1,\ldots,\bar{Z}_m \in \mathrm{T}_{\bar{y}}J^1\pi$ are such that $\mathrm{T}_{\bar{y}}\pi^1\bar{Z}_\alpha = Z_\alpha$.

The restricted Legendre map associated with $\mathcal{L}$ is $\mathcal{FL} := \mu\circ\widetilde{\mathcal{FL}}\colon J^1\pi \to J^1\pi^*$.

In natural coordinates we have:

$$\begin{array}{llll}\widetilde{\mathcal{FL}}^* x^\alpha = x^\alpha & , \quad \widetilde{\mathcal{FL}}^* y^A = y^A & , \quad \widetilde{\mathcal{FL}}^* p_A^\alpha = \dfrac{\partial \pounds}{\partial v_\alpha^A} & , \quad \widetilde{\mathcal{FL}}^* p = \pounds - v_\alpha^A \dfrac{\partial \pounds}{\partial v_\alpha^A} \\ \mathcal{FL}^* x^\alpha = x^\alpha & , \quad \mathcal{FL}^* y^A = y^A & , \quad \mathcal{FL}^* p_A^\alpha = \dfrac{\partial \pounds}{\partial v_\alpha^A} & \end{array}$$

Then, observe that $\widetilde{\mathcal{FL}}^*\Theta = \Theta_{\mathcal{L}}$, and $\widetilde{\mathcal{FL}}^*\Omega = \Omega_{\mathcal{L}}$.

Definition 4 *$(J^1\pi,\Omega_{\mathcal{L}})$ is* regular *(*hyper-regular*) if $\mathcal{F}\mathcal{L}$ is a local (global) diffeomorphism. Elsewhere it is* singular. *(This definition is equivalent to that given above).*

$(J^1\pi,\Omega_{\mathcal{L}})$ is almost-regular if

1. *$\mathcal{P} := \mathcal{F}\mathcal{L}(J^1\pi)$ is a closed submanifold of $J^1\pi^*$ (natural embedding $\jmath_0\colon \mathcal{P}\hookrightarrow J^1\pi^*$).*
2. *$\mathcal{F}\mathcal{L}$ is a submersion onto its image.*
3. *The fibres $\mathcal{F}\mathcal{L}^{-1}(\mathcal{F}\mathcal{L}(\bar{y}))$, $\forall\bar{y}\in J^1\pi$, are connected submanifolds of $J^1\pi$.*

3.2 The (hyper)regular case

The usual way of defining (regular) Hamiltonian systems in field theory consists in considering the bundle $\bar{\tau}\colon J^1\pi^*\to M$ and then giving sections $h\colon J^1\pi^*\to\mathcal{M}\pi$ of the projection μ, which are called *Hamiltonian sections* and carry the physical information of the system. Then we can define the differentiable forms

$$\Theta_h := h^*\Theta\in\mathit{\Omega}^m(J^1\pi^*) \quad , \quad \Omega_h := -\mathrm{d}\Theta_h = h^*\Omega\in\mathit{\Omega}^{m+1}(J^1\pi^*)$$

which are the *Hamilton-Cartan m and $(m+1)$ forms* of $J^1\pi^*$ associated with the Hamiltonian section h. The couple $(J^1\pi^*,\Omega_h)$ is said to be a *Hamiltonian system*.

In a local chart of natural coordinates, a Hamiltonian section is specified by a *local Hamiltonian function* $\mathrm{h}\in\mathrm{C}^\infty(U)$, $U\subset J^1\pi^*$, such that $h(x^\nu,y^A,p^\nu_A)\equiv(x^\nu,y^A,p^\nu_A,p=-\mathrm{h}(x^\gamma,y^B,p^\eta_B))$. Then, the local expressions of the Hamilton-Cartan forms associated with h are

$$\Theta_h = p^\nu_A\mathrm{d}y^A\wedge\mathrm{d}^{m-1}x_\nu - \mathrm{h}\mathrm{d}^m x \quad , \quad \Omega_h = -\mathrm{d}p^\nu_A\wedge\mathrm{d}y^A\wedge\mathrm{d}^{m-1}x_\nu + \mathrm{d}\mathrm{h}\wedge\mathrm{d}^m x$$

Notice that Ω_h is 1-nondegenerate; that is, a multisymplectic form (as a simple calculation in coordinates shows).

Now we want to associate Hamiltonian systems to the Lagrangian ones. First we consider the hyper-regular case (the regular case is analogous, but working locally).

If $(J^1\pi,\Omega_{\mathcal{L}})$ is a hyper-regular Lagrangian system, then we have the diagram

$$\begin{tikzcd} & \mathcal{M}\pi \arrow[d, shift right, "\mu"'] \\ J^1\pi \arrow[ur, "\widetilde{\mathcal{F}\mathcal{L}}"] \arrow[r, "\mathcal{F}\mathcal{L}"] & J^1\pi^* \arrow[u, shift right, "h"'] \end{tikzcd}$$

It is proved [11] that $\tilde{\mathcal{P}} := \widetilde{\mathcal{F}\mathcal{L}}(J^1\pi)$ is a 1-codimensional imbedded submanifold of $\mathcal{M}\pi$ ($\tilde{\jmath}_0\colon\tilde{\mathcal{P}}\hookrightarrow\mathcal{M}\pi$ denotes is the natural embedding), which is transverse to μ, and is diffeomorphic to $J^1\pi^*$. This diffeomorphism is μ^{-1}, when μ is restricted to $\tilde{\mathcal{P}}$, and also coincides with the map $h := \widetilde{\mathcal{F}\mathcal{L}}\circ\mathcal{F}\mathcal{L}^{-1}$, when it is restricted onto its image (which is just $\tilde{\mathcal{P}}$). Thus h and $(J^1\pi^*,\Omega_h)$ are the *Hamiltonian section* and the *Hamiltonian system associated with the hyper-regular Lagrangian system* $(J^1\pi,\Omega_{\mathcal{L}})$, respectively.

Locally, the Hamiltonian section $h(x^\nu,y^A,p^\nu_A)=(x^\nu,y^A,p^\nu_A,p=-\mathrm{h}(x^\gamma,y^B,p^\gamma_B))$ is specified by the *local Hamiltonian function*

$$\mathrm{h} = p^\nu_A(\mathcal{F}\mathcal{L}^{-1})^*v^A_\nu - (\mathcal{F}\mathcal{L}^{-1})^*\pounds$$

Then we have the following local expressions for the corresponding Hamilton-Cartan forms

$$\begin{aligned}\Theta_h &= p^\alpha_A\mathrm{d}y^A\wedge\mathrm{d}^{m-1}x_\alpha - \mathrm{h}\mathrm{d}^m x\\ \Omega_h &= -\mathrm{d}p^\alpha_A\wedge\mathrm{d}y^A\wedge\mathrm{d}^{m-1}x_\alpha + \mathrm{d}\mathrm{h}\wedge\mathrm{d}^m x\end{aligned}$$

and, of course, $\mathcal{FL}^*\Theta_h = \Theta_{\mathcal{L}}$, and $\mathcal{FL}^*\Omega_h = \Omega_{\mathcal{L}}$.

The Hamiltonian field equations can also be derived from a variational principle. In fact:

Definition 5 *Let $(J^1\pi^*, \Omega_h)$ be a Hamiltonian system. Let $\Gamma(M, J^1\pi^*)$ be the set of sections of $\bar{\tau}$. Consider the map*

$$\begin{array}{cccc} \mathbf{H} & : & \Gamma(M, J^1\pi^*) & \longrightarrow & \mathbb{R} \\ & & \psi & \mapsto & \int_M \psi^*\Theta_h \end{array}$$

(where the convergence of the integral is assumed). The variational problem *for this Hamiltonian system is the search for the critical (or stationary) sections of the functional $\mathbf{H}$, with respect to the variations of ψ given by $\psi_t = \sigma_t \circ \psi$, where $\{\sigma_t\}$ is the local one-parameter group of any compact-supported $Z \in \mathfrak{X}^{\mathrm{V}(\bar{\tau})}(J^1\pi^*)$ (where $\mathfrak{X}^{\mathrm{V}(\bar{\tau})}(J^1\pi^*$ denotes the module of $\bar{\tau}$-vertical vector fields in $J^1\pi^*$), that is:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_M \psi_t^*\Theta_h = 0$$

This is the so-called Hamilton-Jacobi principle *of the Hamiltonian formalism.*

The Hamilton-Jacobi principle is equivalent to find distributions $\mathcal{D}$ of $J^1\pi^*$ such that:

1. $\mathcal{D}$ is m-dimensional.
2. $\mathcal{D}$ is $\bar{\tau}$-transverse.
3. $\mathcal{D}$ is integrable (that is, *involutive*).
4. The integral manifolds of $\mathcal{D}$ are the critical sections of the Hamilton-Jacobi principle.

As in the Lagrangian formalism, $\mathcal{D}$ are associated with classes of integrable and $\bar{\tau}$-transverse m-multivector fields $\{\mathcal{X}\} \subset \mathfrak{X}^m(J^1\pi^*)$ or, what is equivalent, with connections in the bundle $\bar{\pi}\colon J^1\pi \to M$, whose expressions are

$$\begin{aligned} \mathcal{X} &= \bigwedge_{\nu=1}^{m} f\left(\frac{\partial}{\partial x^\nu} + F_\nu^A\frac{\partial}{\partial y^A} + G_{A\nu}^\rho\frac{\partial}{\partial p_A^\rho}\right) \quad , \quad (f \in \mathrm{C}^\infty(J^1\pi^*) \text{ non-vanishing}) \\ \nabla &= \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + F_\mu^A\frac{\partial}{\partial y^A} + G_{A\mu}^\rho\frac{\partial}{\partial p_A^\rho}\right) \end{aligned}$$

Then we have:

Theorem 2 *The following assertions on a section $\psi \in \Gamma(M, J^1\pi^*)$ are equivalent:*

1. *ψ is a critical section for the variational problem posed by the Hamilton-Jacobi principle.*
2. *$\psi^*\, i(X)\Omega_h = 0$, $\forall\, X \in \mathfrak{X}(J^1\pi^*)$.*
3. *ψ is an integral section of a class of integrable and $\bar{\tau}$-transverse multivector fields $\{\mathcal{X}_h\} \subset \mathfrak{X}^m(J^1\pi^*)$ satisfying that*
$$i(\mathcal{X}_h)\Omega_h = 0 \quad , \quad \forall\, \mathcal{X}_h \in \{\mathcal{X}_h\} \tag{3}$$
4. *ψ is an integral section of an integrable connection ∇_h in $J^1\pi^*$ satisfying the equation*
$$i(\nabla_h)\Omega_h = (m-1)\Omega_h \tag{4}$$

5. *If* $(U; x^\nu, y^A, p_A^\nu)$ *is a natural system of coordinates in* $J^1\pi^*$*, then* ψ *satisfies the* Hamilton-De Donder-Weyl equations *in* U

$$\frac{\partial(y^A \circ \psi)}{\partial x^\nu} = \frac{\partial \mathrm{h}}{\partial p_A^\nu} \circ \psi \quad ; \quad \frac{\partial(p_A^\nu \circ \psi)}{\partial x^\nu} = -\frac{\partial \mathrm{h}}{\partial y^A} \circ \psi$$

(*Proof*) See, for instance, [17, 22, 23, 52]. ■

$\bar{\tau}$-transverse locally decomposable multivector fields and connections which are solution to the Lagrangian equations (3) and (4) respectively (but not necessarily integrable) are called *Hamilton-De Donder-Weyl multivector fields* and *connections* for $(J^1\pi^*, \Omega_h)$.

The existence of Hamilton-De Donder-Weyl multivector fields and connections for $(J^1\pi^*, \Omega_h)$ is assured, although they are not necessarily integrable.

Finally, we can establish the equivalence between the Lagrangian and Hamiltonian formalisms in the hyper-regular case:

Theorem 3 (Equivalence theorem) *Let* $(J^1\pi, \Omega_\mathcal{L})$ *be a hyper-regular Lagrangian system, and* $(J^1\pi^*, \Omega_h)$ *the associated Hamiltonian system.*

If a section $\phi \in \Gamma(M, E)$ *is a solution to the Lagrangian variational problem (Hamilton principle), then the section* $\psi = \mathcal{FL} \circ j^1\phi \in \Gamma(M, J^1\pi^*)$ *is a solution to the Hamiltonian variational problem (Hamilton-Jacobi principle).*

Conversely, if $\psi \in \Gamma(M, J^1\pi^*)$ *is a solution to the Hamiltonian variational problem, then the section* $\phi = \tau \circ \psi \in \Gamma(M, E)$ *is a solution to the Lagrangian variational problem.*

(*Proof*) See, for instance, [22, 23, 52]. ■

$$\begin{tikzcd}
J^1\pi \arrow[rr, "\mathcal{FL}"] \arrow[dr, "\pi^1"] & & J^1\pi^* \arrow[dl, "\tau"'] \\
& E \arrow[d, shift left, "\pi"] & \\
& M \arrow[uul, "j^1\phi"] \arrow[uur, "\psi"'] \arrow[u, shift left, "\phi"] &
\end{tikzcd}$$

3.3 The almost-regular case

Now, consider the almost-regular case. Let $\tilde{\mathcal{P}} := \widetilde{\mathcal{FL}}(J^1\pi)$, $\mathcal{P} := \mathcal{FL}(J^1\pi)$ (the natural projections are denoted by $\tau_0^1 \colon \mathcal{P} \to E$ and $\bar{\tau}_0^1 := \pi \circ \tau_0^1 \colon \mathcal{P} \to M$), and assume that $\mathcal{P}$ is a fibre bundle over E and M. Denote by $\tilde{\jmath}_0 \colon \tilde{\mathcal{P}} \hookrightarrow \mathcal{M}\pi$ the natural imbedding, and by $\widetilde{\mathcal{FL}}_0$ and $\mathcal{FL}_0$ the restrictions of $\widetilde{\mathcal{FL}}$ and $\mathcal{FL}$ to their images, respectively. We have

$$\begin{tikzcd}
& \tilde{\mathcal{P}} \arrow[rr, "\tilde{\jmath}_0"] \arrow[d, shift left, "\tilde{\mu}"] & & \mathcal{M}\pi \arrow[d, "\mu"] \\
J^1\pi \arrow[ur, "\widetilde{\mathcal{FL}}_0"] \arrow[r, "\mathcal{FL}_0"] & \mathcal{P} \arrow[rr, "\jmath_0"] \arrow[u, shift left, "h_\mathcal{P}"] \arrow[dr, "\bar{\tau}_0"'] & & J^1\pi^* \arrow[dl, "\bar{\tau}"] \\
& & M &
\end{tikzcd}$$

Now, it can be proved that the μ-transverse submanifold $\tilde{\mathcal{P}}$ is diffeomorphic to $\mathcal{P}$ [52]. This diffeomorphism is denoted $\tilde{\mu}\colon \tilde{\mathcal{P}} \to \mathcal{P}$, and it is just the restriction of the projection μ to $\tilde{\mathcal{P}}$. Then, taking $h_{\mathcal{P}} := \tilde{\mu}^{-1}$, we define the Hamilton-Cartan forms

$$\Theta_h^0 = (\tilde{\jmath}_0 \circ h_{\mathcal{P}})^*\Theta \in \mathit{\Omega}^m(\mathcal{P}) \quad , \quad \Omega_h^0 = -\mathrm{d}\Theta_h^0 (\tilde{\jmath}_0 \circ h_{\mathcal{P}})^*\Omega \in \mathit{\Omega}^{m+1}(\mathcal{P})$$

which verify that $\mathcal{FL}_0^*\Omega_h^0 = \Omega_{\mathcal{L}}$. Then $h_{\mathcal{P}}$ is also called a *Hamiltonian section*, and $\mathcal{P}, \Omega_h^0)$ is the *Hamiltonian system* associated with the almost-regular Lagrangian system $(J^1\pi, \Omega_{\mathcal{L}})$. In general, Ω_h^0 is a pre-multisymplectic form and $\mathcal{P}, \Omega_h^0)$ is the *Hamiltonian system associated with the almost-regular Lagrangian system* $(J^1\pi, \Omega_{\mathcal{L}})$.

In this framework, the *Hamilton-Jacobi principle* for $\mathcal{P}, \Omega_h^0)$ is stated like above, and the critical sections $\psi_0 \in \Gamma(M, \mathcal{P})$ can be characterized in an analogous way than in Theorem 2.

If Ω_h^0 is a pre-multisymplectic form, Hamilton-De Donder-Weyl multivector vector fields and connections only exist, in the most favourable cases, in some submanifold $S \hookrightarrow J^1\pi$, and they are not necessarily integrable. As in the Lagrangian case, S can be obtained after applying the suitable constraint algorithm [53]. Then, the equivalence theorem follows in an analogous way than above.

4 Unified Lagrangian-Hamiltonian formalism

4.1 Geometric framework

The *extended* and the *restricted jet-multimomentum bundles* are

$$\mathcal{W} := J^1\pi \times_E \mathcal{M}\pi \quad , \quad \mathcal{W}_r := J^1\pi \times_E J^1\pi^*$$

with natural coordinates $(x^\alpha, y^A, v_\alpha^A, p_A^\alpha, p)$ and $(x^\alpha, y^A, v_\alpha^A, p_A^\alpha)$. We have natural projections (submersions) $\mu_{\mathcal{W}}\colon \mathcal{W} \to \mathcal{W}_r$, and

$$\begin{array}{lllllll} \rho_1\colon \mathcal{W} \to J^1\pi & , & \rho_2\colon \mathcal{W} \to \mathcal{M}\pi & , & \rho_E\colon \mathcal{W} \to E & , & \rho_M\colon \mathcal{W} \to M \\ \rho_1^r\colon \mathcal{W}_r \to J^1\pi & , & \rho_2^r\colon \mathcal{W}_r \to J^1\pi^* & , & \rho_E^r\colon \mathcal{W}_r \to E & , & \rho_M^r\colon \mathcal{W}_r \to M \end{array} \tag{5}$$

Definition 6 *The* coupling m-form *in $\mathcal{W}$, denoted by $\mathcal{C}$, is an m-form along ρ_M which is defined as follows: for every $\bar{y} \in J_y^1E$, with $\bar{\pi}^1(\bar{y}) = \pi(y) = x \in E$, and $\mathbf{p} \in \mathcal{M}_y\pi$, let $w \equiv (\bar{y}, \mathbf{p}) \in \mathcal{W}_y$, then*

$$\mathcal{C}(w) := (\mathrm{T}_x\phi)^*\mathbf{p}$$

where $\phi\colon M \to E$ satisfies that $j^1\phi(x) = \bar{y}$. Then, we denote by $\hat{\mathcal{C}} \in \mathit{\Omega}^m(\mathcal{W})$ the ρ_M-semibasic form associated with $\mathcal{C}$.

The canonical m-form $\Theta_{\mathcal{W}} \in \mathit{\Omega}^m(\mathcal{W})$ *is defined as $\Theta_{\mathcal{W}} := \rho_2^*\Theta$, and is ρ_E-semibasic. The* canonical $(m+1)$-form *is the pre-multisymplectic form $\Omega_{\mathcal{W}} := -\mathrm{d}\Theta_{\mathcal{W}} = \rho_1^*\Omega \in \mathit{\Omega}^{m+1}(\mathcal{W})$.*

There exists $\hat{C} \in \mathrm{C}^\infty(\mathcal{W})$ such that $\hat{\mathcal{C}} = \hat{C}(\rho_M^*\omega)$, and $\hat{\mathcal{C}}(w) = (p + p_A^\alpha v_\alpha^A)\mathrm{d}^m x$.

Local expressions of $\Theta_{\mathcal{W}}$ and $\Omega_{\mathcal{W}}$ are the same than for Θ and Ω.

Let $\hat{\mathcal{L}} := \rho_1^*\mathcal{L} \in \mathit{\Omega}^m(\mathcal{W})$, and $\hat{\mathcal{L}} = \hat{L}(\rho_M^*\omega)$, with $\hat{L} = \rho_1^*L \in \mathrm{C}^\infty(\mathcal{W})$. We define the *Hamiltonian submanifold* $\jmath_0\colon \mathcal{W}_0 \hookrightarrow \mathcal{W}$ by

$$\mathcal{W}_0 := \{w \in \mathcal{W} \mid \hat{\mathcal{L}}(w) = \hat{\mathcal{C}}(w)\}$$

The constraint function defining $\mathcal{W}_0$ is

$$\hat{C} - \hat{L} = p + p_A^\alpha v_\alpha^A - \hat{L}(x^\nu, y^B, v_\nu^B) = 0$$

There are projections which are the restrictions to $\mathcal{W}_0$ of the projections (5):

$$\begin{tikzcd}
 & & J^1\pi & & \\
\mathcal{W}_0 \arrow[urr, "\rho_1^0"] \arrow[rr, "\jmath_0"] \arrow[drr, "\rho_2^0"] \arrow[ddrr, "\hat{\rho}_2^0"'] & & \mathcal{W} \arrow[u, "\rho_1"] \arrow[rr, "\mu_{\mathcal{W}}"] \arrow[d, "\rho_2"] & & \mathcal{W}_r \arrow[ull, "\rho_1^r"'] \arrow[dll, "\rho_2^r"'] \arrow[ddll, "\hat{\rho}_2^r"] \\
 & & \mathcal{M}\pi \arrow[d, "\mu"] & & \\
 & & J^1\pi^* & &
\end{tikzcd}$$

$(x^\alpha, y^A, v^A_\alpha, p^\alpha_A)$ are local coordinates in $\mathcal{W}_0$, and

$$\begin{array}{lcl}
\rho_1^0(x^\alpha, y^A, v^A_\alpha, p^\alpha_A) = (x^\alpha, y^A, v^A_\alpha) & , & \jmath_0(x^\alpha, y^A, v^A_\alpha, p^\alpha_A) = (x^\alpha, y^A, v^A_\alpha, p^\alpha_A, L - v^A_\alpha p^\alpha_A) \\
\hat{\rho}_2^0(x^\alpha, y^A, v^A_\alpha, p^\alpha_A) = (x^\alpha, y^A, p^\alpha_A) & , & \rho_2^0(x^\alpha, y^A, v^A_\alpha, p^\alpha_A) = (x^\alpha, y^A, p^\alpha_A, L - v^A_\alpha p^\alpha_A)
\end{array}$$

It is proved that $\mathcal{W}_0$ is a 1-codimensional $\mu_{\mathcal{W}}$-transversal submanifold of $\mathcal{W}$, diffeomorphic to $\mathcal{W}_r$. As a consequence, $\mathcal{W}_0$ induces a *Hamiltonian section* of $\mu_{\mathcal{W}}$, $\hat{h}\colon \mathcal{W}_r \to \mathcal{W}$, which is locally specified by giving the local *Hamiltonian function* $\hat{H} = -\hat{L} + p^\alpha_A v^A_\alpha$; that is, $\hat{h}(x^\alpha, y^A, v^A_\alpha, p^\alpha_A) = (x^\alpha, y^A, v^A_\alpha, p^\alpha_A, -\hat{H})$. From $\hat{h}$ we recover a Hamiltonian section $\tilde{h}\colon \mathcal{P} \to \mathcal{M}\pi$ defined by $\tilde{h}([\mathbf{p}]) = (\rho_2 \circ \hat{h})[(\rho_2^r)^{-1}(\jmath([\mathbf{p}]))]$, $\forall [\mathbf{p}] \in \mathcal{P}$.

$$\begin{tikzcd}
\tilde{\mathcal{P}} \arrow[rr, "\tilde{\jmath}"'] \arrow[d, shift left, "\tilde{\mu}"] & & \mathcal{M}\pi \arrow[d, "\mu"] & & \mathcal{W} \arrow[ll, "\rho_2"'] \\
\mathcal{P} \arrow[u, shift left, "\tilde{\mu}^{-1}"] \arrow[rr, "\jmath"'] \arrow[urr, "\tilde{h}"'] & & J^1\pi^* & & \mathcal{W}_r \arrow[ll, "\rho_2^r"] \arrow[u, "\hat{h}"']
\end{tikzcd}$$

(For hyper-regular systems we have $\tilde{\mathcal{P}} = \mathcal{M}\pi$ and $\mathcal{P} = J^1\pi^*$).

We define the forms $\Theta_0 := \jmath_0^*\Theta_{\mathcal{W}} = \rho_2^{0*}\Theta \in \Omega^m(\mathcal{W}_0)$, and $\Omega_0 := \jmath_0^*\Omega_{\mathcal{W}} = \rho_2^{0*}\Omega \in \Omega^{m+1}(\mathcal{W}_0)$, whose local expressions are

$$\begin{aligned}
\Theta_0 &= (L - p^\alpha_A v^A_\alpha)\mathrm{d}^m x + p^\alpha_A \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\alpha \\
\Omega_0 &= \mathrm{d}(p^\alpha_A v^A_\alpha - L) \wedge \mathrm{d}^m x - \mathrm{d}p^\alpha_A \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\alpha
\end{aligned}$$

$(\mathcal{W}_0, \Omega_0)$ (equiv. $(\mathcal{W}_r, \hat{h}^*\Omega_0)$) is a pre-multisymplectic Hamiltonian system.

4.2 Field equations

A *Lagrange-Hamilton problem* consists in finding sections $\psi_0 \in \Gamma(M, \mathcal{W}_0)$ such that

$$\psi_0^*\, i(Y_0)\Omega_0 = 0 \quad , \quad \forall Y_0 \in \mathfrak{X}(\mathcal{W}_0) \tag{6}$$

Taking $Y_0 \in \mathfrak{X}^{\mathrm{V}(\hat{\rho}_2^0)}(\mathcal{W}_0)$ we get the *first constraint submanifold* $\jmath_1\colon \mathcal{W}_1 \hookrightarrow \mathcal{W}_0$,

$$\mathcal{W}_1 = \{(\bar{y}, \mathbf{p}) \in \mathcal{W}_0 \mid \ i(V_0)(\Omega_0)_{(\bar{y},\mathbf{p})} = 0, \text{ for every } V_0 \in \mathrm{V}(\hat{\rho}_2^0)\}$$

and sections solution to (6) take values on it. $\mathcal{W}_1$ is defined by $p_A^\alpha = \dfrac{\partial L}{\partial v_\alpha^A}$, hence

$$\mathcal{W}_1 = \{(\bar{y}, \widetilde{\mathcal{FL}}(\bar{y})) \in \mathcal{W} \ \mid \ \bar{y} \in J^1\pi\}$$

and $\mathcal{W}_1$ is diffeomorphic to $J^1\pi$.

Theorem 4 (see diagram (7)). *Let $\psi_0\colon M \to \mathcal{W}_0$ be a section fulfilling equation (6), then $\psi_0 = (\psi_{\mathcal{L}}, \psi_{\mathcal{H}}) = (\psi_{\mathcal{L}}, \widetilde{\mathcal{FL}} \circ \psi_{\mathcal{L}})$, where $\psi_{\mathcal{L}} = \rho_1^0 \circ \psi_0$, and:*

1. *$\psi_{\mathcal{L}}$ is the canonical lift of the projected section $\phi = \rho_E^0 \circ \psi_0\colon M \to E$ (that is, $\psi_{\mathcal{L}}$ is a holonomic section).*
2. *$\psi_{\mathcal{L}} = j^1\phi$ is a solution to the Lagrangian problem, and $\mu \circ \psi_{\mathcal{H}} = \mu \circ \widetilde{\mathcal{FL}} \circ \psi_{\mathcal{L}} = \mathcal{FL} \circ j^1\phi$ is a solution to the Hamiltonian problem.*

   Conversely, for every section $\phi\colon M \to E$ such that $j^1\phi$ is a solution to the Lagrangian problem (and hence $\mathcal{FL} \circ j^1\phi$ is a solution to the Hamiltonian problem) we have that $\psi_0 = (j^1\phi, \widetilde{\mathcal{FL}} \circ j^1\phi)$, is a solution to (6).

(*Proof*) See [18] and [54]. ■

Thus, equation (6) gives equations of three different classes:

1. Algebraic equations, determining $\mathcal{W}_1 \hookrightarrow \mathcal{W}_0$, where the sections solution take their values. These are the *primary Hamiltonian constraints*, and generate, by $\hat{\rho}_2^0$ projection, the primary constraints of the Hamiltonian formalism for singular Lagrangians.
2. Differential equations, forcing the sections solution ψ_0 to be holonomic.
3. The Euler-Lagrange equations.

$$\begin{tikzcd}
 & \mathcal{W} \arrow[ddl, "\rho_1"'] \arrow[dr, "\rho_2"] & & \\
 & \mathcal{W}_0 \arrow[u, "\jmath_0"] \arrow[r, "\rho_2^0"] \arrow[dl, "\rho_1^0"'] & \mathcal{M}\pi & \\
J^1\pi \arrow[dr, "\pi^1"] & \mathcal{W}_1 \arrow[l, "\rho_1^1"'] \arrow[r, "\rho_2^1"] \arrow[u, "\jmath_1"] \arrow[d, "\rho_E^1"'] & J^1\pi^* \arrow[dl, "\tau^1"] & \mathcal{M}\pi \\
 & E & & \\
 & M \arrow[u, "\phi"] \arrow[uu, shift left, "\psi_1"'] \arrow[uuu, shift left=2, "\psi_0"'] \arrow[uul, "\psi_{\mathcal{L}} = j^1\phi"] \arrow[uurr, "\psi_{\mathcal{H}} = \widetilde{\mathcal{FL}} \circ j^1\phi"'] & &
\end{tikzcd} \tag{7}$$

Field equations in the unified formalism can also be stated in terms of multivector fields and connections in $\mathcal{W}_0$. In fact, the problem of finding sections solution to (6) can be formulated equivalently as follows: finding a distribution D_0 of $\mathrm{T}(\mathcal{W}_0)$ such that it is integrable (that is, *involutive*), m-dimensional, ρ_M^0-transverse, and the integral manifolds of D_0 are the sections solution to the above equations. (Note that we do not ask them to be lifting of π-sections; that is, the holonomic condition). This is equivalent to stating that the sections solution to this problem are the integral sections of one of the following equivalent elements:

- A class of integrable and ρ_M^0-transverse m-multivector fields $\{X_0\} \subset \mathfrak{X}^m(\mathcal{W}_0)$ satisfying that
$$i(X_0)\Omega_0 = 0 \quad , \quad \text{for every } X_0 \in \{X_0\}$$
- An integrable connection ∇_0 in $\rho_M^0 \colon \mathcal{W}_0 \to M$ such that
$$i(\nabla_0)\Omega_0 = (m-1)\Omega_0$$

Locally decomposable and ρ_M^0-transverse multivector fields and orientable connections which are solutions of these equations are called *Lagrange-Hamiltonian multivector fields* and *jet fields* for $(\mathcal{W}_0, \Omega_0)$. Euler-Lagrange and hamilton-De Donder-Weyl multivector fields can be recovered from these Lagrange-Hamiltonian multivector fields (see [18]).

A Appendix

A.1 Multisymplectic manifolds

Definition 7 *Let $\mathcal{M}$ be a differentiable manifold, and $\Omega \in \mathit{\Omega}^k(\mathcal{M})$ ($1 < k \leq \dim \mathcal{M}$).*

Ω *is a* multisymplectic form*, and then* $(\mathcal{M}, \Omega)$ *is a* multisymplectic manifold*, if*

1. $\Omega \in Z^k(\mathcal{M})$ *(it is closed).*
2. Ω *is* 1*-nondegenerate; that is, for every* $p \in \mathcal{M}$ *and* $X_p \in \mathrm{T}_p\mathcal{M}$*,* $i(X_p)\Omega_p = 0 \Leftrightarrow X_p = 0$.

If Ω *is closed and* 1*-degenerate then it is a* pre-multisymplectic form*, and* $(\mathcal{M}, \Omega)$ *is a* pre-multisymplectic manifold*.*

Multisymplectic manifolds of degree $k = 2$ are the usual symplectic manifolds, and manifolds with a distinguished volume form are multisymplectic manifolds of degree its dimension. Other examples of multisymplectic manifolds are provided by compact semisimple Lie groups equipped with the canonical cohomology 3-class, symplectic 6-dimensional Calabi-Yau manifolds with the canonical 3-class, etc. There are no multisymplectic manifolds of degrees 1 or $\dim \mathcal{M} - 1$ because $\ker \Omega$ is nonvanishing in both cases.

Another very important kind of multisymplectic manifold is the *multicotangent bundle* of a manifold Q, $\Lambda^k(\mathrm{T}^*Q)$, that is, the bundle of k-forms in Q. This bundle is endowed with a canonical k-form $\Theta \in \mathit{\Omega}^k(\Lambda^k(\mathrm{T}^*Q)$, and then $\Omega := -\mathrm{d}\Theta \in \mathit{\Omega}^{k+1}(\Lambda^k(\mathrm{T}^*Q)$ is a 1-nondegenerate form. Then the couple $(\Lambda^k(\mathrm{T}^*Q), \Omega)$ is a multisymplectic manifold.

A local classification of multisymplectic forms can be done only for particular cases [55].

A.2 Multivector fields

(See [21] for details).

Let $\mathcal{M}$ be a n-dimensional differentiable manifold. Sections of $\Lambda^m(\mathrm{T}\mathcal{M})$ are called *m-multivector fields* in $\mathcal{M}$ (they are the contravariant skew-symmetric tensors of order m in $\mathcal{M}$). We denote by $\mathfrak{X}^m(\mathcal{M})$ the set of m-multivector fields in $\mathcal{M}$. Then, $\mathcal{X} \in \mathfrak{X}^m(\mathcal{M})$ is *locally decomposable* if, for every $p \in \mathcal{M}$, there is an open neighbourhood $U_p \subset \mathcal{M}$ and $X_1, \dots, X_m \in \mathfrak{X}(U_p)$ such that $\mathcal{X}|_{U_p} = X_1 \wedge \dots \wedge X_m$.

A non-vanishing $\mathcal{X} \in \mathfrak{X}^m(\mathcal{M})$ and a m-dimensional distribution $\mathcal{D} \subset \mathrm{T}\mathcal{M}$ are *locally associated* if there exists a connected open set $U \subseteq \mathcal{M}$ such that $\mathcal{X}|_U$ is a section of $\Lambda^m \mathcal{D}|_U$. If $\mathcal{X}, \mathcal{X}' \in \mathfrak{X}^m(\mathcal{M})$ are non-vanishing multivector fields locally associated with the same distribution $\mathcal{D}$, on the same connected open set U, then there exists a non-vanishing function $f \in \mathrm{C}^\infty(U)$ such that $\mathcal{X}'|_U = f\mathcal{X}$. This fact defines an equivalence relation in the set of non-vanishing m-multivector fields in $\mathcal{M}$, whose equivalence classes will be denoted by $\{\mathcal{X}\}_U$. Then there is a one-to-one correspondence between the m-dimensional orientable distributions $\mathcal{D}$ in $\mathrm{T}\mathcal{M}$ and the equivalence classes $\{\mathcal{X}\}_{\mathcal{M}}$ of non-vanishing, locally decomposable m-multivector fields in $\mathcal{M}$.

A non-vanishing, locally decomposable multivector field $\mathcal{X} \in \mathfrak{X}^m(\mathcal{M})$ is said to be *integrable* (resp. *involutive*) if its associated distribution is integrable (resp. involutive). If $\mathcal{X} \in \mathfrak{X}^m(\mathcal{M})$ is integrable (resp. involutive), then so is every other in its equivalence class $\{\mathcal{X}\}$, and all of them have the same integral manifolds. Moreover, *Frobenius theorem* allows us to say that a non-vanishing and locally decomposable multivector field is integrable if, and only if, it is involutive.

If $\pi \colon \mathcal{M} \to M$ is a fiber bundle, we are interested in the case where the integral manifolds of integrable multivector fields in $\mathcal{M}$ are sections of π. Thus, $\mathcal{X} \in \mathfrak{X}^m(\mathcal{M})$ is said to be *π-transverse* if, at every point $y \in \mathcal{M}$, $(i(\mathcal{X})(\pi^*\beta))_y \neq 0$, for every $\beta \in \Omega^m(M)$ with $\omega(\pi(y)) \neq 0$. Then, if $\mathcal{X} \in \mathfrak{X}^m(\mathcal{M})$ is integrable, it is π-transverse if, and only if, its integral manifolds are local sections of $\pi \colon \mathcal{M} \to M$. Finally, it is clear that classes of locally decomposable and π-transverse multivector fields $\{\mathcal{X}\} \subseteq \mathfrak{X}^m(\mathcal{M})$ are in one-to-one correspondence with orientable Ehresmann connection forms ∇ in $\pi \colon \mathcal{M} \to M$. This correspondence is characterized by the fact that the horizontal subbundle associated with ∇ is the distribution associated with $\{\mathcal{X}\}$. In this correspondence, classes of integrable locally decomposable and π-transverse m multivector fields correspond to flat orientable Ehresmann connections.

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